INVERSE PROBLEM OF GROUP ANALYSIS FOR AUTONOMOUS DIFFERENTIAL SYSTEMS[1]

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Abstract

Full set of autonomous completely solvable differential systems of equations in total differentials is built by basis of infinitesimal operators, universal invariant, and structure constants of admited multiparametric Lie group (abelian and non-abelian).

Key words: differential system, Lie group.

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Problem statement. Consider an autonomous completely solvable system of equations in total differentials

$$dx = f(x)\,dt, \quad dx = \mathrm{colon}\,(dx_1, \ldots, dx_n), \quad dt = \mathrm{colon}\,(dt_1, \ldots, dt_m), \tag{1}$$

on a domain Ω from the phase space $\mathbb{K}^n$ (real $\mathbb{R}^n$ or complex $\mathbb{C}^n$) and let the entries of the $(n \times m)$-matrix $f(x) = \|f_{ij}(x)\|$ for all $x \in \Omega$ be holomorphic functions $f_{ij}\colon \Omega \to \mathbb{K},\ i = 1, \ldots, n,\ j = 1, \ldots, m.$

The linear differential operators of first order

$$\mathfrak{F}_j(x) = \sum_{i=1}^{n} f_{ij}(x)\,\partial_{x_i} \quad \text{for all } x \in \Omega, \quad j = 1, \ldots, m,$$

are autonomous operators of differentiation by virtue of system (1) [1, p. 20; 6; 7].

An operational criterion of completely solvability on the domain Ω of system (1) is represented via Poisson brackets as the system of identities [1, pp. 17 – 25; 6]

$$\big[\mathfrak{F}_j(x), \mathfrak{F}_\zeta(x)\big] = \mathfrak{O} \quad \text{for all } x \in \Omega, \quad j = 1, \ldots, m,\ \zeta = 1, \ldots, m.$$

Suppose a q-parameter Lie group of transformation G_q, $1 \leqslant q \leqslant n$, [8 – 11] has $(n-k)$-cylindrical infinitesimal operators (coordinates of these operators not depends on $n-k$ dependent variables [1, pp. 106 – 167; 7])

$$\mathfrak{G}_l(x) = \sum_{i=1}^{n} \mathrm{g}_{li}({}^k x)\,\partial_{x_i} \quad \text{for all } x \in \Omega, \quad l = 1, \ldots, q, \quad {}^k x = (x_1, \ldots, x_k),\ 1 \leqslant k \leqslant n, \tag{2}$$

and a basis of absolute invariants

$$I\colon x \to \big(I_1(x), \ldots, I_{n-q}(x)\big) \quad \text{for all } x \in \Omega. \tag{3}$$

[1] The basic results of this paper have been published in the monograph "*Integrals of differential systems*", Grodno, 2006, 447 p. [1, pp. 152 – 160] and in the journals [2 – 5]: *Differential Equations*, Vol. 30, 1994, No. 5, 899-901; *Vestsi Nats. Akad. Navuk Belarusi*, Ser. 2, 1995, No. 2, 46-49; *Proceedings of the scientific seminars POMI*, Vol. 232, 1996, 86-89; *Dokl. Akad. Nauk Belarusi*, Vol. 50, 2006, No. 3, 15-19.

Besides, the Poisson brackets

$$\big[\mathfrak{G}_l(x), \mathfrak{G}_s(x)\big] = \sum_{p=1}^{q} c_{lsp}\,\mathfrak{G}_p(x) \quad \text{for all } \ x \in \Omega, \quad l = 1, \dots, q, \quad s = 1, \dots, q, \tag{4}$$

where c_{lsp}, $l=1,\dots,q$, $s=1,\dots,q$, $p=1,\dots,q$, are the structure constants of Lie group G_q.

Assume that the coordinate functions g_{li}, $l = 1,\dots,q$, $i = 1,\dots,n$, of the infinitesimal operators (2) and the absolute invariants I_τ, $\tau = 1,\dots,n-q$, of the basis (3) are holomorphic on the domain Ω. Furthermore, the infinitesimal operators (2) are not holomorphically linearly bound [1, p. 11; 6] on the domain Ω.

In this paper, the following inverse problem of group analysis for differential systems is considered: to select from the set off all completely solvable differential systems (1) the systems that admit a Lie group G_q with the infinitesimal operators (2), the universal invariant (3), and the structure constants c_{lsp} from the operator identities (4).

Classes of differential systems that admit an non-abelian Lie group. Suppose the completely solvable autonomous system of equations in total differentials (1) admits a Lie group G_q. Then, the Poisson brackets

$$\big[\mathfrak{G}_l(x), \mathfrak{F}_j(x)\big] = \mathfrak{O} \quad \text{for all } \ x \in \Omega, \quad l = 1, \dots, q, \quad j = 1, \dots, m. \tag{5}$$

Lemma. *Suppose the linear differential operators of first order* (2) *are not holomorphically linearly bound on the domain* Ω. *Then the following conditions are true*:

i) *the complete linear homogeneous system of partial differential equations*

$$\mathfrak{G}_l(x)\, u = 0, \quad l = 1, \dots, q, \quad q < n, \tag{6}$$

admits $(n-q)$*-parameter abelian Lie group* $\widetilde{G}_{n-q}$;

ii) *infinitesimal operators of the abelian Lie group* $\widetilde{G}_{n-q}$ *and the operators* $\mathfrak{G}_l$, $l=1,\dots,q$, *are commutative and they are not holomorphically linearly bound on the domain* Ω.

Proof (analogous statement for linear homogeneous partial differential equation was proved in [9, p. 109]). Since the system (6) is complete [1, p. 38; 6; 12, p. 70], we see that integral basis of this system is $n-q$ first integrals [1, p. 50]

$$F_\xi \colon x \to F_\xi(x) \quad \text{for all } \ x \in \Omega, \quad \xi = 1, \dots, n-q.$$

Then, the substitution

$$\nu_\xi = F_\xi(x), \ \ \xi = 1, \dots, n-q, \qquad \nu_\zeta = \Phi_\zeta(x), \ \ \zeta = n-q+1, \dots, n,$$

where the functions Φ_ζ, $\zeta = n-q+1,\dots,n$, are first integrals of the complete linear nonhomogeneous partial differential systems [12, p. 101]

$$\mathfrak{G}_l(x)\, u = \delta_{l,\zeta-n+q}, \quad l = 1, \dots, q, \ \ \zeta = n-q+1, \dots, n, \ \ (\delta_{ls} \text{ is Kronecker symbol}),$$

reduces the system (6) to the complete linear homogeneous partial differential system

$$\partial_{\nu_\zeta} u = 0, \quad \zeta = n-q+1, \dots, n.$$

This system admits the $n-q$ differential operators ∂_{ν_ξ}, $\xi = 1,\dots,n-q$. At the same time these operators are not holomorphically linearly bound on the space $\mathbb{K}^{n-q}$ and they admits $(n-q)$-parameter abelian Lie group.

Consequently the system (6) admits $(n-q)$-parameter abelian Lie group $\widetilde{G}_{n-q}$.

The operators ∂_{ν_i}, $i = 1,\dots,n$, are commutative and these operators are not holomorphically linearly bound on the space $\mathbb{K}^n$. In addition, the property of symmetry for Poisson brackets and the property of linear bound (or not linear bound) for operators are conserva-

tion under transformation. This implies that: infinitesimal operators of the abelian Lie group $\widetilde{G}_{n-q}$ and the operators $\mathfrak{G}_l,\ l = 1,\dots,q,$ are commutative; infinitesimal operators of the abelian Lie group $\widetilde{G}_{n-q}$ and the operators $\mathfrak{G}_l,\ l = 1,\dots,q,$ are not holomorphically linearly bound on the domain Ω. The lemma is proved.

Using the operators (2) admitted by the system (1), we can build the $(n-k)$-cylindrical [7] linear homogeneous system of partial differential equations

$$\mathfrak{G}_l(x)\, u = 0, \quad l = 1,\dots,q, \quad q \leqslant n. \tag{7}$$

By the conditions (4), it follows that the system (7) is complete [1, p. 38; 6] on Ω.

From Lemma it follows that the complete system (7) admits an $(n-q)$-parameter abelian group $\widetilde{G}_{n-q}$. Let the linear differential operators of first order

$$\mathfrak{G}_\tau(x) = \sum_{i=1}^{n} \mathrm{g}_{\tau i}(x)\, \partial_{x_i} \quad \text{for all } x \in \Omega, \quad \tau = q+1,\dots,n,$$

be infinitesimal operators of the abelian Lie group $\widetilde{G}_{n-q}$.

By Lemma, the Poisson brackets

$$\big[\mathfrak{G}_i(x), \mathfrak{G}_\tau(x)\big] = \mathfrak{O} \quad \text{for all } x \in \Omega, \quad i = 1,\dots,n, \quad \tau = q+1,\dots,n. \tag{8}$$

Note also that the linear differential operators $\mathfrak{G}_i,\ i = 1,\dots,n,$ on the domain Ω are not holomorphically linearly bound.

Let the operators $\mathfrak{G}_i,\ i = 1,\dots,n,$ be a basis. Then the operators $\mathfrak{F}_j,\ j = 1,\dots,m,$ induced by the system (1) have the expansions

$$\mathfrak{F}_j(x) = \sum_{i=1}^{n} \psi_{ji}(x)\, \mathfrak{G}_i(x) \quad \text{for all } x \in \Omega, \quad j = 1,\dots,m, \tag{9}$$

where $\psi_{ji}\colon \Omega \to \mathbb{K},\ j = 1,\dots,m,\ i = 1,\dots,n,$ are holomorphic functions.

Using the expansions (9) and the commutator identities (5) and (8), we obtain necessary and sufficient conditions that the completely solvable system (1) admits the Lie group G_q:

$$\big[\mathfrak{G}_l(x), \mathfrak{F}_j(x)\big] = \sum_{i=1}^{n} \mathfrak{G}_l \psi_{ji}(x)\, \mathfrak{G}_i(x) + \sum_{i=1}^{q}\sum_{p=1}^{q} c_{lpi}\, \psi_{jp}(x)\, \mathfrak{G}_i(x) \quad \text{for all } x \in \Omega,$$

$$l = 1,\dots,q, \quad j = 1,\dots,m.$$

Since the commutator identities (5) hold and the operators $\mathfrak{G}_i,\ i = 1,\dots,n,$ are basis, we have

$$\begin{gathered}\mathfrak{G}_l\, \psi_{j\theta}(x) + \sum_{p=1}^{q} c_{lp\theta}\, \psi_{jp}(x) = 0 \quad \text{for all } x \in \Omega, \quad l = 1,\dots,q, \quad j = 1,\dots,m, \quad \theta = 1,\dots,q,\\ \mathfrak{G}_l\, \psi_{j\tau}(x) = 0 \quad \text{for all } x \in \Omega, \quad l = 1,\dots,q, \quad j = 1,\dots,m, \quad \tau = q+1,\dots,n.\end{gathered} \tag{10}$$

From this system of identities it follows that the scalar functions $\psi_{j\tau},\ j = 1,\dots,m,$ $\tau = q+1,\dots,n,$ are absolute invariants of Lie group G_q. Therefore,

$$\psi_{j\tau}(x) = \varphi_{j\tau}(I(x)) \quad \text{for all } x \in \Omega, \quad j = 1,\dots,m, \quad \tau = q+1,\dots,n,$$

where the functions $\varphi_{j\tau}\colon W \to \mathbb{K},\ j = 1,\dots,m,\ \tau = q+1,\dots,n,$ are holomorphic on the domain W of the space $\mathbb{K}^{n-q}$.

Taking into account the system of identities (10), we obtain the scalar functions $\psi_{j\theta}$, $j=1,\ldots,m,\ \theta=1,\ldots,q,$ are first integrals of the linear non-homogeneous system of partial differential equations

$$\mathfrak{G}_l(x)\,\psi_{j\theta}+\sum_{p=1}^{q}c_{lp\theta}\,\psi_{jp}=0,\quad l=1,\ldots,q,\quad j=1,\ldots,m,\quad \theta=1,\ldots,q. \tag{11}$$

By the identities (4) and (8), it follows that the system (11) is complete on the domain Ω. Suppose this system has the first integrals

$$\psi_{jl}\colon x\to\ \psi_{jl}(x)\quad\text{for all }\ x\in\Omega,\quad j=1,\ldots,m,\quad l=1,\ldots,q.$$

Then, we obtain the representation

$$\mathfrak{F}_j(x)=\sum_{l=1}^{q}\psi_{jl}(x)\,\mathfrak{G}_l(x)+\sum_{\tau=q+1}^{n}\varphi_{j\tau}(I(x))\,\mathfrak{G}_\tau(x)\quad\text{for all }\ x\in\Omega,\quad j=1,\ldots,m,$$

and the following statement.

Theorem 1. *The autonomous completely solvable system of equations in total differentials* (1) *admits the q-parameter Lie group G_q with $(n-k)$-cylindrical infinitesimal operators* (2), *the universal invariant* (3), *and the structure constants from the representations* (4) *if and only if this differential system has the form*

$$dx_i=\sum_{j=1}^{m}\Bigg(\sum_{l=1}^{q}\psi_{jl}(x)\,\mathrm{g}_{li}(\overset{k}{x})+\sum_{\tau=q+1}^{n}\varphi_{j\tau}(I(x))\,\mathrm{g}_{\tau i}(x)\Bigg)\,dt_j,\quad i=1,\ldots,n, \tag{12}$$

where the holomorphic functions $\psi_{jl}\colon\Omega\to\mathbb{K},\ j=1,\ldots,m,\ l=1,\ldots,q,$ are first integrals of the complete linear non-homogeneous system of partial differential equations (11), *the holomorphic functions $\mathrm{g}_{\tau i}\colon\Omega\to\mathbb{K},\ \tau=q+1,\ldots,n,\ i=1,\ldots,n,$ are the coordinates of the infinitesimal operators $\mathfrak{G}_\tau,\ \tau=q+1,\ldots,n,$ of $(n-q)$-parameter abelian Lie group $\widetilde{G}_{n-q}$ admitted by the $(n-k)$-cylindrical complete linear homogeneous system of partial differential equations* (7), *and the holomorphic on the domain $W\subset\mathbb{K}^{n-q}$ functions $\varphi_{j\tau}\colon W\to\mathbb{K},\ j=1,\ldots,m,\ \tau=q+1,\ldots,n,$ such that the conditions of completely solvability on the domain Ω for system* (12) *hold.*

Classes of differential systems that admit an abelian Lie group. Suppose the q-parameter Lie group G_q is abelian. Then in the representation (4) the structure constants

$$c_{lsp}=0,\quad l=1,\ldots,q,\quad s=1,\ldots,q,\quad p=1,\ldots,q.$$

Also, from the system of identities (10) it follows that the functions $\psi_{ji},\ j=1,\ldots,m,$ $i=1,\ldots,n,$ are absolute invariants of the q-parameter abelian Lie group G_q. Therefore,

$$\psi_{ji}(x)=\varphi_{ji}(I(x))\quad\text{for all }\ x\in\Omega,\quad j=1,\ldots,m,\quad i=1,\ldots,n,$$

where the functions $\varphi_{ji}\colon W\to\mathbb{K},\ j=1,\ldots,m,\ i=1,\ldots,n,$ are holomorphic on the domain W from the space $\mathbb{K}^{n-q}$.

Then,

$$\mathfrak{F}_j(x)=\sum_{l=1}^{q}\varphi_{jl}(I(x))\,\mathfrak{G}_l(x)+\sum_{\tau=q+1}^{n}\varphi_{j\tau}(I(x))\,\mathfrak{G}_\tau(x)\quad\text{for all }\ x\in\Omega,\quad j=1,\ldots,m,$$

and we have

Theorem 2. *The autonomous completely solvable system of equations in total differentials* (1) *admits the* q*-parameter abelian Lie group* G_q *with* $(n-k)$*-cylindrical infinitesimal operators* (2) *and the universal invariant* (3) *if and only if this system has the form*

$$dx_i = \sum_{j=1}^{m}\Big(\sum_{l=1}^{q}\varphi_{jl}(I(x))\,\mathrm{g}_{li}({}^{k}x) + \sum_{\tau=q+1}^{n}\varphi_{j\tau}(I(x))\,\mathrm{g}_{\tau i}(x)\Big)\,dt_j, \quad i=1,\dots,n,$$

where the holomorphic functions $\mathrm{g}_{\tau i}\colon \Omega\to\mathbb{K}$, $\tau=q+1,\dots,n$, $i=1,\dots,n$, *are the coordinates of the infinitesimal operators* $\mathfrak{G}_\tau$, $\tau=q+1,\dots,n$, *of* $(n-q)$*-parameter abelian Lie group* $\widetilde{G}_{n-q}$ *admitted by the* $(n-k)$*-cylindrical complete linear homogeneous system of partial differential equations* (7), *and the holomorphic on the domain* $W\subset\mathbb{K}^{n-q}$ *functions* $\varphi_{ji}\colon W\to\mathbb{K}$, $j=1,\dots,m$, $i=1,\dots,n$, *such that the conditions of completely solvability on the domain* Ω *for this built differential system hold.*

Example 1. Consider the q-parameter abelian Lie group of transformations

$$x_l \to e^{\alpha_l}x_l, \ \ l=1,\dots,q, \quad x_\tau\to x_\tau, \ \ \tau=q+1,\dots,n, \ \ q<n, \tag{13}$$

with $(n-q)$-cylindrical infinitesimal operators

$$\mathfrak{G}_l(x) = x_l\,\partial_{x_l} \ \text{ for all } \ x\in\mathbb{K}^n, \quad l=1,\dots,q, \tag{14}$$

and the basis of absolute invariants

$$I\colon x\to (x_{q+1},\dots,x_n) \ \text{ for all } \ x\in\mathbb{K}^n. \tag{15}$$

By Theorem 2, we have the following

Proposition 1. *The completely solvable autonomous system of equations in total differentials* (1) *admits the* q*-parameter abelian Lie group of transformations* (13) *with the infinitesimal operators* (14) *and the universal invariant* (15) *if and only if this system has the form*

$$dx_i = \sum_{j=1}^{m}\Big(\sum_{l=1}^{q}\delta_{il}\,x_l\,\varphi_{jl}(x_{q+1},\dots,x_n) + \sum_{\tau=q+1}^{n}\mathrm{g}_{\tau i}(x)\,\varphi_{j\tau}(x_{q+1},\dots,x_n)\Big)dt_j, \ i=1,\dots,n, \tag{16}$$

where the holomorphic functions $\mathrm{g}_{\tau i}\colon\Omega\to\mathbb{K}$, $\tau=q+1,\dots,n$, $i=1,\dots,n$, *are the coordinates of the linear differential operators of first order* $\mathfrak{G}_\tau(x)=\sum\limits_{i=1}^{n}\mathrm{g}_{\tau i}(x)\,\partial_{x_i}$ *for all* $x\in\Omega$, $\tau=q+1,\dots,n$ (*the set of operators* $\mathfrak{G}_\tau$ *and* (14) *is commutative and these operators aren't holomorphic linearly bound on the domain* $\Omega\subset\mathbb{K}^n$), *and the holomorphic on the domain* Ω^{n-q} *from the space* $\mathbb{K}^{n-q}$ *functions* φ_{ji}, $j=1,\dots,m$, $i=1,\dots,n$, *such that the conditions of completely solvability on the domain* Ω *for the system* (16) *hold.*

The set of the linear differential operators of first order

$$\mathfrak{G}_\tau(x) = x_\tau\,\partial_{x_\tau} \ \text{ for all } \ x\in\mathbb{K}^n, \quad \tau=q+1,\dots,n,$$

and the infinitesimal operators (14) is commutative and these operators aren't holomorphic linearly bound on the space $\mathbb{K}^n$. Then, from Proposition 1, we obtain

Proposition 2. *An system of equations in total differentials*

$$dx_i = \sum_{j=1}^{m} x_i\varphi_{ji}(x_{q+1},\dots,x_n)\,dt_j, \quad i=1,\dots,n,$$

where φ_{ji}, $j=1,\dots,m$, $i=1,\dots,n$, *are holomorphic functions on a domain* Ω^{n-q} *from the space* $\mathbb{K}^{n-q}$ *such that the Frobenius conditions hold*

$$\sum_{\nu=1}^{n} x_\nu \varphi_{\mu\nu}(x_{q+1},\ldots,x_n)\,\partial_{x_\nu}\big(x_i\varphi_{ji}(x_{q+1},\ldots,x_n)\big) =$$

$$= \sum_{\nu=1}^{n} x_\nu \varphi_{j\nu}(x_{q+1},\ldots,x_n)\,\partial_{x_\nu}\big(x_i\varphi_{\mu i}(x_{q+1},\ldots,x_n)\big) \quad \textit{for all } \ x\in\Omega, \quad \Omega\subset\mathbb{K}^n,$$

$$i=1,\ldots,n, \quad j=1,\ldots,m, \quad \mu=1,\ldots,m,$$

admits the q-parameter abelian Lie group of transformations (13) *with the infinitesimal operators* (14) *and the universal invariant* (15).

Example 2. The one-parameter Lie group of dilatations of space

$$x_i \to e^{\alpha}x_i, \quad i=1,\ldots,n, \tag{17}$$

has the infinitesimal operator

$$\mathfrak{G}_1(x)=\sum_{i=1}^{n} x_i\,\partial_{x_i} \quad \text{for all } \ x\in\mathbb{K}^n \tag{18}$$

and the universal invariant

$$I\colon x\to\Big(\frac{x_1}{x_n},\ldots,\frac{x_{n-1}}{x_n}\Big) \quad \text{for all } \ x\in\Omega, \quad \Omega\subset\mathbb{K}^n. \tag{19}$$

By Theorem 2 (under $q=1,\ m=1$), we have

Proposition 3. *An autonomous ordinary differential system of the n-th order admits the one-parameter Lie group of dilatations* (17) *with the infinitesimal operator* (18) *and the universal invariant* (19) *if and only if this system has the form*

$$\frac{dx_i}{dt}=x_i\,\varphi_1\Big(\frac{x_1}{x_n},\ldots,\frac{x_{n-1}}{x_n}\Big)+\sum_{\tau=2}^{n}\mathrm{g}_{\tau i}(x)\,\varphi_\tau\Big(\frac{x_1}{x_n},\ldots,\frac{x_{n-1}}{x_n}\Big), \quad i=1,\ldots,n,$$

where the holomorphic functions $\mathrm{g}_{\tau i}\colon\Omega\to\mathbb{K},\ \tau=2,\ldots,n,\ i=1,\ldots,n,$ *are the coordinates of the linear differential operators of first order* $\mathfrak{G}_\tau(x)=\sum\limits_{i=1}^{n}\mathrm{g}_{\tau i}(x)\,\partial_{x_i}$ *for all* $x\in\Omega$, $\tau=2,\ldots,n$ (*the set of operators* $\mathfrak{G}_\tau$ *and* (18) *is commutative and these operators aren't holomorphic linearly bound on the domain* $\Omega\subset\mathbb{K}^n$), *and the functions* $\varphi_i\colon W\to\mathbb{K},\ i=1,\ldots,n,$ *are holomorphic on the domain* W *from the space* $\mathbb{K}^{n-1}$.

The set of the linear differential operators of first order

$$\mathfrak{G}_\tau(x)=x_\tau\,\partial_{x_\tau} \quad \text{for all } \ x\in\mathbb{K}^n, \quad \tau=2,\ldots,n,$$

and the infinitesimal operators (18) is commutative and these operators aren't holomorphic linearly bound on the space $\mathbb{K}^n$. Then, from Proposition 3, we have

Proposition 4. *An autonomous ordinary differential system*

$$\frac{dx_i}{dt}=x_i\varphi_i\Big(\frac{x_1}{x_n},\ldots,\frac{x_{n-1}}{x_n}\Big), \quad i=1,\ldots,n,$$

where the functions $\varphi_i\colon W\to\mathbb{K},\ i=1,\ldots,n,$ *are holomorphic on the domain* W *from the space* $\mathbb{K}^{n-1}$, *admits the one-parameter Lie group of dilatations* (17) *with the infinitesimal operator* (18) *and the universal invariant* (19).

Example 3. Consider the one-parameter Lie group of rotation of real plane

$$u=x\cos\alpha-y\sin\alpha, \quad v=x\sin\alpha+y\cos\alpha \tag{20}$$

with the group parameter $\alpha \in \mathbb{R}$. This Lie group has the infinitesimal operator

$$\mathfrak{G}(x,y) = -y\,\partial_x + x\,\partial_y \quad \text{for all } (x,y) \in \mathbb{R}^2 \tag{21}$$

and the universal invariant

$$I\colon (x,y) \to x^2 + y^2 \quad \text{for all } (x,y) \in \mathbb{R}^2. \tag{22}$$

By Theorem 2 (under $q = 1,\, m = 1,\, n = 2$), we have

Proposition 5. *An autonomous ordinary differential system of second order admits the one-parameter Lie group of rotation of phase plane* (20) *with the infinitesimal operator* (21) *and the universal invariant* (22) *if and only if this system has the form*

$$\frac{dx}{dt} = -y\,\varphi_1(x^2+y^2) + a_x(x,y)\,\varphi_2(x^2+y^2), \quad \frac{dy}{dt} = x\,\varphi_1(x^2+y^2) + a_y(x,y)\,\varphi_2(x^2+y^2),$$

where the holomorphic functions $a_x\colon \Omega \to \mathbb{R}$ *and* $a_y\colon \Omega \to \mathbb{R}$ *are the coordinates of the linear differential operator of first order* $\mathfrak{A}(x,y) = a_x(x,y)\,\partial_x + a_y(x,y)\,\partial_y$ *for all* $(x,y) \in \Omega$ (*the operator* $\mathfrak{A}$ *is an differential operator such that* $\mathfrak{A}$ *is commutative with the infinitesimal operator* (21) *and these operators aren't holomorphic linearly bound on the domain* $\Omega \subset \mathbb{R}^2$), *and sections of functions* $\varphi_1\colon [0;\, +\infty) \to \mathbb{R}$ *and* $\varphi_2\colon [0;\, +\infty) \to \mathbb{R}$ *are holomorphic on a set* $W \subset [0;\, +\infty)$.

The linear differential operator of first order

$$\mathfrak{A}(x,y) = x\,\partial_x + y\,\partial_y \quad \text{for all } (x,y) \in \mathbb{R}^2$$

and the infinitesimal operator (21) aren't holomorphic linearly bound on the plane $\mathbb{R}^2$ and they are commutative operators. Then, using Proposition 5, we get

Proposition 6. *An autonomous ordinary differential system*

$$\frac{dx}{dt} = -y\,\varphi_1(x^2+y^2) + x\,\varphi_2(x^2+y^2), \quad \frac{dy}{dt} = x\,\varphi_1(x^2+y^2) + y\,\varphi_2(x^2+y^2), \tag{23}$$

where sections of functions $\varphi_1\colon [0;\, +\infty) \to \mathbb{R}$ *and* $\varphi_2\colon [0;\, +\infty) \to \mathbb{R}$ *are holomorphic on* $W \subset [0;\, +\infty)$, *admits the one-parameter Lie group of rotation of phase plane* (20) *with the infinitesimal operator* (21) *and the universal invariant* (22).

Let us remark that a differential equation of the first order that admit the Lie group of rotation (20) was considered in [13, p. 149]. This equation is the equation of trajectories for the autonomous ordinary differential system (23).

There exists (Theorem 4.1 in [14, p. 23]) the formal change of variables

$$u = x + \sum_{k=2}^{\infty} U_k(x,y), \qquad v = y + \sum_{k=2}^{\infty} V_k(x,y), \tag{24}$$

where $U_k\colon \mathbb{R}^2 \to \mathbb{R}$ and $V_k\colon \mathbb{R}^2 \to \mathbb{R}$ are homogeneous polynomials of degrees $\deg U_k(x,y) = \deg V_k(x,y) = k,\ k = 2,3,\ldots,$ such that the differential system

$$\frac{du}{dt} = -v - \sum_{k=2}^{\infty} P_k(u,v), \qquad \frac{dv}{dt} = u + \sum_{k=2}^{\infty} Q_k(u,v), \tag{25}$$

where $P_k\colon \mathbb{R}^2 \to \mathbb{R}$ and $Q_k\colon \mathbb{R}^2 \to \mathbb{R}$ are homogeneous polynomials of degrees $\deg P_k(u,v) = \deg Q_k(u,v) = k,\ k = 2,3,\ldots,$ reduces to the system (23) with $\varphi_1(0) = 1,\ \varphi_2(0) = 0$.

Therefore, we have

Proposition 7. *Suppose the autonomous ordinary differential system* (25) *has an equilibrum state with purely imaginary characteristic roots. Then, there exists the formal change of dependent variables* (24) *such that the system* (25) *reduces to an autonomous differential system that admit the one-parameter Lie group of rotation of phase plane.*

The autonomous differential system (23) that admit the one-parameter Lie group of rotation of phase plane (20) is the normal form [15] of the autonomous differential system (25) with an equilibrum state as purely imaginary characteristic roots.

Example 4. Consider the one-parameter Lie group of Lorentz transformations of real plane

$$u = x\cosh\alpha + y\sinh\alpha, \quad v = x\sinh\alpha + y\cosh\alpha \tag{26}$$

with the group parameter $\alpha \in \mathbb{R}$. This Lie group has the infinitesimal operator

$$\mathfrak{G}(x,y) = y\,\partial_x + x\,\partial_y \quad \text{for all} \ \ (x,y) \in \mathbb{R}^2 \tag{27}$$

and the universal invariant

$$I\colon (x,y) \to y^2 - x^2 \quad \text{for all} \ \ (x,y) \in \mathbb{R}^2. \tag{28}$$

By Theorem 2 (under $q=1,\ m=1,\ n=2$), we have

Proposition 8. *An autonomous ordinary differential system of second order admits the one-parameter Lie group of Lorentz transformations of phase plane* (26) *with the infinitesimal operator* (27) *and the universal invariant* (28) *if and only if this system has the form*

$$\frac{dx}{dt} = y\,\varphi_1(y^2-x^2) + a_x(x,y)\,\varphi_2(y^2-x^2), \quad \frac{dy}{dt} = x\,\varphi_1(y^2-x^2) + a_y(x,y)\,\varphi_2(y^2-x^2),$$

where the holomorphic functions $a_x\colon \Omega \to \mathbb{R}$ *and* $a_y\colon \Omega \to \mathbb{R}$ *are the coordinates of the linear differential operator of first order* $\mathfrak{A}(x,y) = a_x(x,y)\,\partial_x + a_y(x,y)\,\partial_y$ *for all* $(x,y)\in\Omega$ (*the operator* $\mathfrak{A}$ *is an differential operator such that* $\mathfrak{A}$ *is commutative with the infinitesimal operator* (27) *and these operators aren't holomorphic linearly bound on the domain* $\Omega \subset \mathbb{R}^2$), *and functions* $\varphi_1\colon W \to \mathbb{R}$ *and* $\varphi_2\colon W \to \mathbb{R}$ *are holomorphic on a set* $W \subset \mathbb{R}$.

The linear differential operator of first order

$$\mathfrak{A}(x,y) = x\,\partial_x + y\,\partial_y \quad \text{for all} \ \ (x,y) \in \mathbb{R}^2$$

and the infinitesimal operator (27) aren't holomorphic linearly bound on the plane $\mathbb{R}^2$ and they are commutative operators. Then, using Proposition 8, we get

Proposition 9. *An autonomous ordinary differential system*

$$\frac{dx}{dt} = y\,\varphi_1(y^2-x^2) + x\,\varphi_2(y^2-x^2), \quad \frac{dy}{dt} = x\,\varphi_1(y^2-x^2) + y\,\varphi_2(y^2-x^2),$$

where functions $\varphi_1\colon W \to \mathbb{R}$ *and* $\varphi_2\colon W \to \mathbb{R}$ *are holomorphic on* $W \subset \mathbb{R}$, *admits the one-parameter Lie group of Lorentz transformations of phase plane* (26) *with the infinitesimal operator* (27) *and the universal invariant* (28).

Example 5. Consider the one-parameter Lie group of projective transformations of real plane

$$u = \frac{x}{1-\alpha x}\,, \qquad v = \frac{y}{1-\alpha x} \tag{29}$$

with the group parameter $\alpha \in \mathbb{R}$. This Lie group has the infinitesimal operator

$$\mathfrak{G}(x,y) = x^2\,\partial_x + xy\,\partial_y \quad \text{for all} \ \ (x,y) \in \mathbb{R}^2 \tag{30}$$

and the universal invariant

$$I\colon (x,y) \to \frac{x}{y} \quad \text{for all } (x,y) \in \{(x,y)\colon y \neq 0\}. \tag{31}$$

By Theorem 2 (under $q=1,\ m=1,\ n=2$), we get

Proposition 10. *An autonomous ordinary differential system of second order admits the one-parameter Lie group of projective transformations of phase plane* (29) *with the infinitesimal operator* (30) *and the universal invariant* (31) *if and only if this system has the form*

$$\frac{dx}{dt} = x^2\,\varphi_1\Big(\frac{x}{y}\Big) + a_x(x,y)\,\varphi_2\Big(\frac{x}{y}\Big), \qquad \frac{dy}{dt} = xy\,\varphi_1\Big(\frac{x}{y}\Big) + a_y(x,y)\,\varphi_2\Big(\frac{x}{y}\Big),$$

where the holomorphic functions $a_x\colon \Omega \to \mathbb{R}$ and $a_y\colon \Omega \to \mathbb{R}$ are the coordinates of the linear differential operator of first order $\mathfrak{A}(x,y) = a_x(x,y)\,\partial_x + a_y(x,y)\,\partial_y$ for all $(x,y) \in \Omega$ (the operator $\mathfrak{A}$ is an differential operator such that $\mathfrak{A}$ is commutative with the infinitesimal operator (30) *and these operators aren't holomorphic linearly bound on the domain $\Omega \subset \mathbb{R}^2$), and functions $\varphi_1\colon W \to \mathbb{R}$ and $\varphi_2\colon W \to \mathbb{R}$ are holomorphic on a set $W \subset \mathbb{R}$.*

The linear differential operator of first order

$$\mathfrak{A}(x,y) = xy\,\partial_x + (x+y^2)\,\partial_y \quad \text{for all } (x,y) \in \mathbb{R}^2$$

and the infinitesimal operator (30) aren't holomorphic linearly bound on the plane $\mathbb{R}^2$ and they are commutative operators. Then, using Proposition 10, we have

Proposition 11. *An autonomous ordinary differential system*

$$\frac{dx}{dt} = x^2\,\varphi_1\Big(\frac{x}{y}\Big) + xy\,\varphi_2\Big(\frac{x}{y}\Big), \qquad \frac{dy}{dt} = xy\,\varphi_1\Big(\frac{x}{y}\Big) + (x+y^2)\,\varphi_2\Big(\frac{x}{y}\Big),$$

where functions $\varphi_1\colon W \to \mathbb{R}$ and $\varphi_2\colon W \to \mathbb{R}$ are holomorphic on $W \subset \mathbb{R}$, admits the one-parameter Lie group of projective transformations of phase plane (29) *with the infinitesimal operator* (30) *and the universal invariant* (31)*.*

Example 6. Consider the one-parameter Lie group of nonhomogeneous stretches of real plane

$$u = e^{\alpha}\,x, \qquad v = e^{k\alpha}\,y \tag{32}$$

with the group parameter $\alpha \in \mathbb{R}$ and the coefficient $k \in \mathbb{R}$. This Lie group has the infinitesimal operator

$$\mathfrak{G}(x,y) = x\,\partial_x + ky\,\partial_y \quad \text{for all } (x,y) \in \mathbb{R}^2 \tag{33}$$

and the universal invariant

$$I\colon (x,y) \to \frac{x^k}{y} \quad \text{for all } (x,y) \in \{(x,y)\colon y \neq 0\}. \tag{34}$$

By Theorem 2 (under $q=1,\ m=1,\ n=2$), we obtain

Proposition 12. *An autonomous ordinary differential system of second order admits the one-parameter Lie group of nonhomogeneous stretches of phase plane* (32) *with the infinitesimal operator* (33) *and the universal invariant* (34) *if and only if this system has the form*

$$\frac{dx}{dt} = x\,\varphi_1\Big(\frac{x^k}{y}\Big) + a_x(x,y)\,\varphi_2\Big(\frac{x^k}{y}\Big), \qquad \frac{dy}{dt} = ky\,\varphi_1\Big(\frac{x^k}{y}\Big) + a_y(x,y)\,\varphi_2\Big(\frac{x^k}{y}\Big),$$

where the holomorphic functions $a_x\colon \Omega \to \mathbb{R}$ and $a_y\colon \Omega \to \mathbb{R}$ are the coordinates of the

linear differential operator of first order $\mathfrak{A}(x,y) = a_x(x,y)\,\partial_x + a_y(x,y)\,\partial_y$ *for all* $(x,y)\in\Omega$ (*the operator* $\mathfrak{A}$ *is an differential operator such that* $\mathfrak{A}$ *is commutative with the infinitesimal operator* (33) *and these operators aren't holomorphic linearly bound on the domain* $\Omega\subset\mathbb{R}^2$), *and functions* $\varphi_1\colon W\to\mathbb{R}$ *and* $\varphi_2\colon W\to\mathbb{R}$ *are holomorphic on a set* $W\subset\mathbb{R}$.

The linear differential operator of first order

$$\mathfrak{A}(x,y) = kx\,\partial_x + y\,\partial_y \quad \text{for all } (x,y)\in\mathbb{R}^2$$

and the infinitesimal operator (33) aren't holomorphic linearly bound on the plane $\mathbb{R}^2$ and they are commutative operators. Then, using Proposition 12, we obtain

Proposition 13. *An autonomous ordinary differential system*

$$\frac{dx}{dt} = x\,\varphi_1\Big(\frac{x^k}{y}\Big) + kx\,\varphi_2\Big(\frac{x^k}{y}\Big), \qquad \frac{dy}{dt} = ky\,\varphi_1\Big(\frac{x^k}{y}\Big) + y\,\varphi_2\Big(\frac{x^k}{y}\Big),$$

where functions $\varphi_1\colon W\to\mathbb{R}$ *and* $\varphi_2\colon W\to\mathbb{R}$ *are holomorphic on* $W\subset\mathbb{R}$, *admits the one-parameter Lie group of nonhomogeneous stretches of phase plane* (32) *with the infinitesimal operator* (33) *and the universal invariant* (34).

Example 7. Consider the one-parameter Lie group of Galilean transformations of real plane

$$u = x + \alpha y, \quad v = y \tag{35}$$

with the group parameter $\alpha\in\mathbb{R}$. This Lie group has the infinitesimal operator

$$\mathfrak{G}(x,y) = y\,\partial_x \quad \text{for all } (x,y)\in\mathbb{R}^2 \tag{36}$$

and the universal invariant

$$I\colon (x,y)\to\ y \quad \text{for all } (x,y)\in\mathbb{R}^2. \tag{37}$$

By Theorem 2 (under $q=1,\ m=1,\ n=2$), we have

Proposition 14. *An autonomous ordinary differential system of second order admits the one-parameter Lie group of Galilean transformations of phase plane* (35) *with the infinitesimal operator* (36) *and the universal invariant* (37) *if and only if this system has the form*

$$\frac{dx}{dt} = y\,\varphi_1(y) + a_x(x,y)\,\varphi_2(y), \qquad \frac{dy}{dt} = a_y(x,y)\,\varphi_2(y),$$

where the holomorphic functions $a_x\colon\Omega\to\mathbb{R}$ *and* $a_y\colon\Omega\to\mathbb{R}$ *are the coordinates of the linear differential operator of first order* $\mathfrak{A}(x,y) = a_x(x,y)\,\partial_x + a_y(x,y)\,\partial_y$ *for all* $(x,y)\in\Omega$ (*the operator* $\mathfrak{A}$ *is an differential operator such that* $\mathfrak{A}$ *is commutative with the infinitesimal operator* (36) *and these operators aren't holomorphic linearly bound on the domain* $\Omega\subset\mathbb{R}^2$), *and functions* $\varphi_1\colon W\to\mathbb{R}$ *and* $\varphi_2\colon W\to\mathbb{R}$ *are holomorphic on a set* $W\subset\mathbb{R}$.

The linear differential operator of first order

$$\mathfrak{A}(x,y) = x\,\partial_x + y\,\partial_y \quad \text{for all } (x,y)\in\mathbb{R}^2$$

and the infinitesimal operator (36) aren't holomorphic linearly bound on the plane $\mathbb{R}^2$ and they are commutative operators. Then, using Proposition 14, we obtain

Proposition 15. *An autonomous ordinary differential system*

$$\frac{dx}{dt} = y\,\varphi_1(y) + x\,\varphi_2(y), \qquad \frac{dy}{dt} = y\,\varphi_2(y),$$

where functions $\varphi_1\colon W\to\mathbb{R}$ *and* $\varphi_2\colon W\to\mathbb{R}$ *are holomorphic on* $W\subset\mathbb{R}$, *admits the one-parameter Lie group of Galilean transformations of phase plane* (35) *with the infinitesimal operator* (36) *and the universal invariant* (37).

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